GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Fall Quarter 1998

Applied Mathematics
EXAM AREA

Assigned Number (DO NOT SIGN YOUR NAME)

- **Please sign your name on the back of this page—**

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M.E. Ph.D. Qualifying Examination
Applied Mathematics

Control Number: ______________

Fall 1998

Instructions: Do four of the five following problems. Your answers should be as complete as possible. Extra sheets of paper are available upon request.

1. Consider the following system of equations:

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\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 3 \\
2x_1 - x_2 + x_3 &= 6, \\
3x_1 + x_2 - kx_3 &= 4
\end{aligned}
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where k is a constant.

(a) What is the matrix representation of this system of equations in the form $Ax = b$?

(b) Set $k = 1$ and find x_1, x_2, and x_3 using Gaussian elimination.

(c) For what value of k will the homogeneous system of equations $Ax = 0$ have a non-trivial solution?

(d) For this same value of k, what is the corresponding characteristic polynomial for the eigenvalues of the matrix A? What does the result of part (c) say about this characteristic polynomial?

M.E. Ph.D. Qualifying Examination

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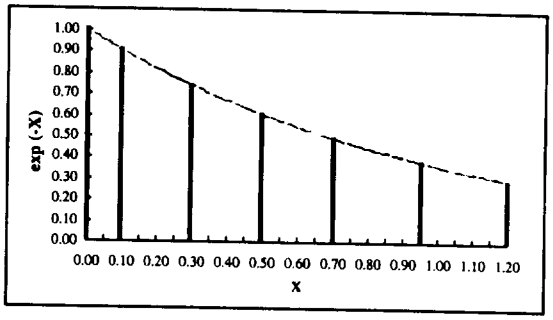
2. The function $f(x) = \exp(-x)$ was used to generate the following table of unequally-spaced data:

x	*f(x)*
0	1
0.1	0.9048
0.3	0.7408
0.5	0.6065
0.7	0.4966
0.95	0.3867
1.2	0.3012



(a) Evaluate the integral from $a = 0$ to $b = 1.2$ analytically.

(b) Evaluate the integral using a combination of the trapezoidal and Simpson's rules. Use Simpson's rules whenever possible to obtain the highest accuracy.

(c) What is the percent error of the answer obtained in part (b)?

Newton-Cotes closed integration formulas. The step size is h = (b - a)/n.

Segments (n)	Points	Name	Formula
1	2	Trapezoidal rule	$(b-a)\dfrac{f(x_0)+f(x_1)}{2}$
2	3	Simpson's 1/3 rule	$(b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$
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M.E. Ph.D. Qualifying Examination

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3. The following boundary value problem describes wave motion on a rectangular membrane in the *xy*-plane that is clamped on two opposite sides ($x = 0, a$) and free on the other two sides ($y = 0, b$):

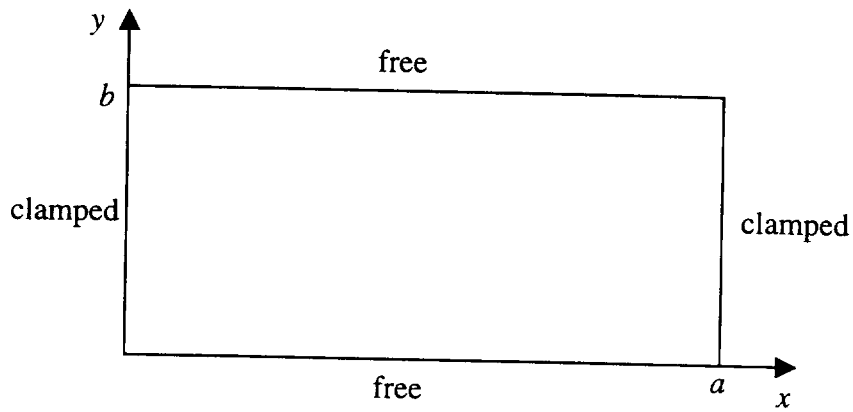
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

and the boundary counditions:

$$u(x = 0, y, t) = u(x = a, y, t) = 0, \qquad \text{and} \qquad u_y(x, y = 0, t) = u_y(x, y = b, t) = 0.$$

Here, $u(x, y, t)$ is the displacement normal to the surface, t is time, c is a constant representing the speed of propagation of the waves, and $u_y = \partial u / \partial y$.

Find the solution of the above PDE *with boundary conditions* by the method of separation of variables. Note that your result should be general in terms of the time integration.



M.E. Ph.D. Qualifying Examination
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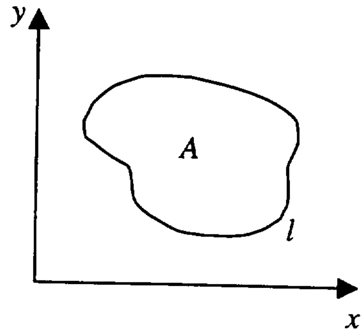
4. Green's theorem states:

$$\oint_l M(x,y)dx + N(x,y)dy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy,$$

where A is an area in the xy-plane and l is its perimeter.



Stokes's theorem states:

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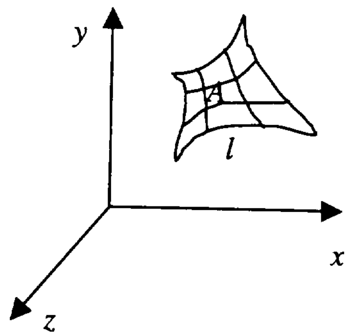

Show the relationship between the above two theorems.

5. A spring-mass system under an external sinusoidal force is described by the following initial value problem:

$$m\ddot{y} + ky = A\sin(bt), \quad y(0) = \dot{y}(0) = 0.$$

Here, m, k, A, and b are all strickly positive constants and $\ddot{y}(t) = d^2y(t)/dt^2$. Solve this problem using the Laplace transform method. In your work, consider the two cases (1) $k/m \neq b^2$ and (2) $k/m = b^2$.

Use the following Laplace transform table, where a>0 is a constant.

$$L[e^{-at}] = \frac{1}{s+a}$$

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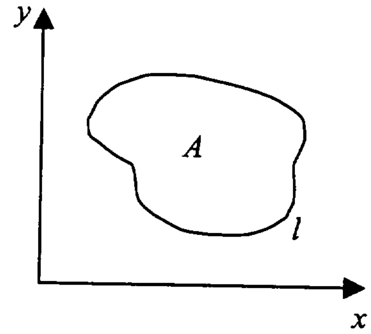

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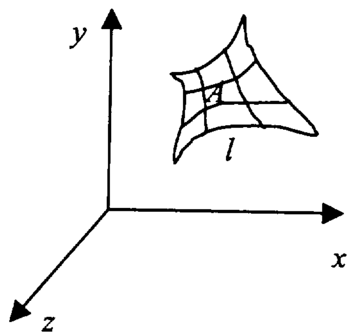

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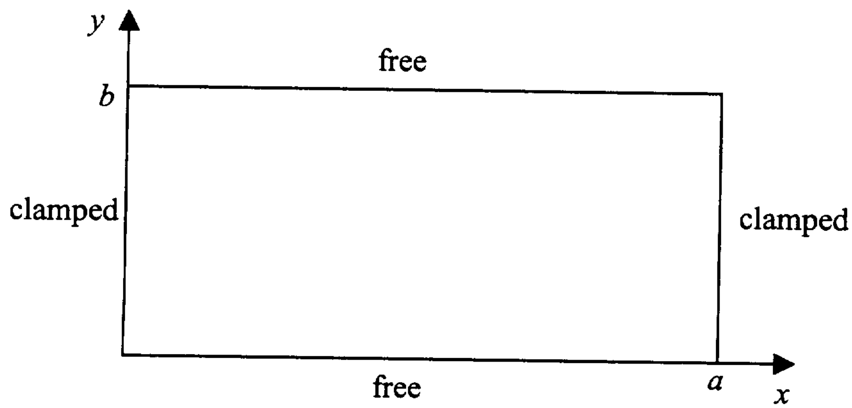

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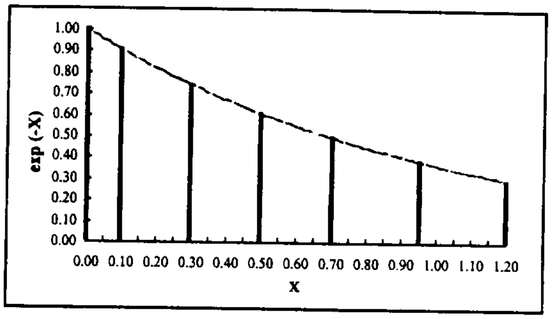

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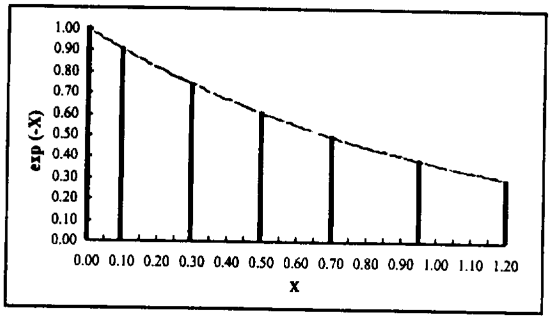

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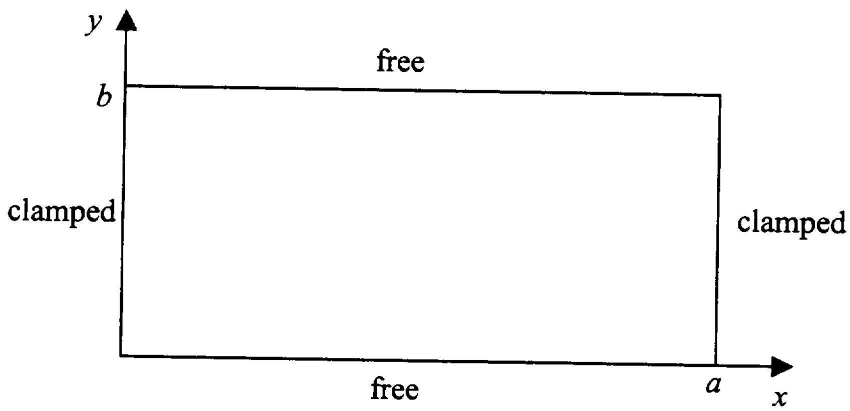

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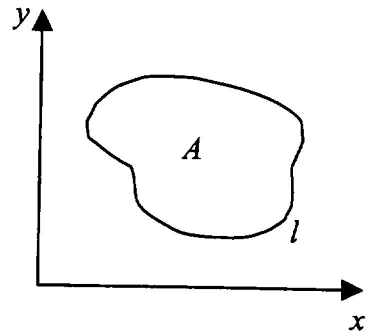

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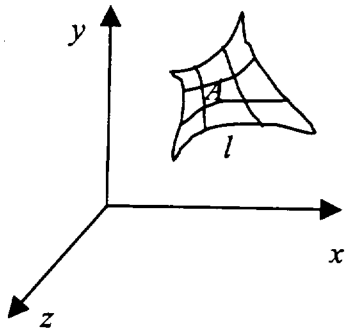

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