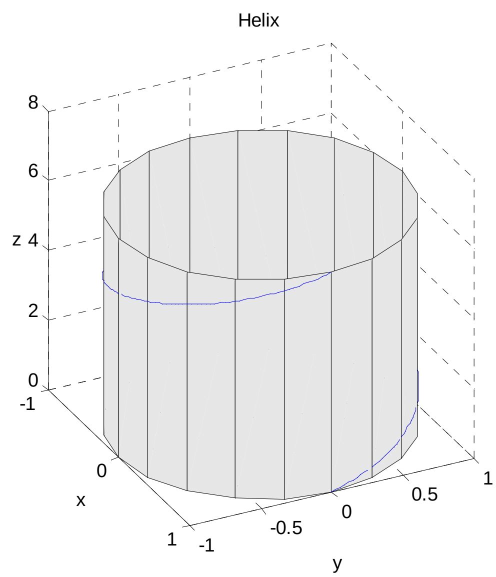
1. Consider the boundary value problem on the domain V

$$\nabla^2 u(\mathbf{x}) + f(\mathbf{x}) = 0 \text{, for } \mathbf{x} \in V$$
$$u(\mathbf{x})\big|_{\mathbf{x}\in S} = g(\mathbf{x}) \text{ ,}$$

where S is the boundary of V and $f(\mathbf{x})$ and $g(\mathbf{x})$ are given functions. Please show that the solution to the above boundary value problem is unique.

2. The figure below shows a cylinder of radius one and length 2π and a helix. The helix starts at the point $(x, y, z) = (1, 0, 0)$ and in one revolution climbs up to the point $(x, y, z) = (1, 0, 2\pi)$. One property of a helix is that the projection of the tangent vector onto the z-axis is constant so that it rises uniformly as it progresses around the cylinder.
   a) Find an appropriate parameterization of the helix.
   b) Find an expression for the unit tangent vector $\vec{t}$ to the helix and compute $\vec{t} \cdot \vec{k}$, where $\vec{k}$ is the unit vector in the z-direction.
   c) Find the length of the helix.



3. (a) One way to compute e^x is to use a Taylor series. Consider the evaluation of e^{-20} and e^{20} using this approach as shown below:

$$e^{-20} = 1 + (-20) + \frac{(-20)^2}{2!} + \frac{(-20)^3}{3!} + \cdots,$$

$$e^{20} = 1 + (20) + \frac{(20)^2}{2!} + \frac{(20)^3}{3!} + \cdots.$$

Which one do you expect to produce more accurate results and why?

(b) For the one that is less accurate, could you suggest a better, i.e., quicker and more accurate, algorithm to compute it?

(Hint: The first 30 terms in the Taylor series of e^{-20} calculated on a computer with a precision of 15 decimal places are shown below:

0:	1.00000000000000	15:	-25058226.1164271
1:	-20.0000000000000	16:	31322782.6455340
2:	200.000000000000	17:	-36850331.5241576
3:	-1333.00000000000	18:	40944813.9157307
4:	6666.00000000000	19:	-43099804.1218218
5:	-26666.6666666667	20:	43099804.1218218
6:	88888.8888888889	21:	-41047432.4969731
7:	-253966.253968253	22:	37315847.7245210
8:	634920.634920635	23:	-32448563.2387139
9:	-1410934.74426808	24:	27040469.3655949
10:	2821869.48853616	25:	-21632375.4924760
11:	-5130671.79733846	26:	16640288.8403661
12:	8551119.66223077	27:	-12326139.8817527
13:	-13155568.7111243	28:	8804385.62982334
14:	18793669.5873204	29:	-6071990.08953334

The results for e^{-20} and e^{20}, accurate to 15 decimal places, are

$e^{-20} = 2.06115362243856 \times 10^{-9}$ and $e^{20} = 4.851651954097903 \times 10^{8}$.)

(c) Solve the following integral equation using elementary numerical methods.

$$y(t) = y(0) + \int_0^t y(t)dt, \qquad t \in [0, T]$$

$$y(0) = y_0$$

4. Consider a circle $x_1^2 + x_2^2 = 1$ in a two-dimensional plane. Suppose that each point (x_1, x_2) is linearly transformed to another point (y_1, y_2) via a matrix A as follows:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find the points on the original circle that, after the transformation, will lie at the maximum and minimum distance from the origin of the two-dimensional plane compared to all the other points.

b) What is the mathematical representation of the circle after the transformation? Sketch it.

c) Based on the result in (b), suggest a matrix B such that after the transformation By, all points of (b) will lie at an equal distance $k(>0)$ from the origin.