Instructions

Please complete all **4** problems attached.

Problem 1

The surface of a toroid is given by the parameterization:

$$x = (a + b\cos t)\cos s$$
$$y = (a + b\cos t)\sin s$$
$$z = b\sin t$$

for $0 \le s \le 2\pi$ and $0 \le t \le 2\pi$. Recall that the surface area is given by the formula:

$$A = \iint_S \left\| \mathbf{T}_s \times \mathbf{T}_t \right\| ds dt$$

where $\mathbf{T}_s$ and $\mathbf{T}_s$ are the tangent vectors with respect to the parameterized coordinates *s* and *t*. What is the surface area of a toroid where $a = 2$ and $b = 3$?

Problem 2

Let $\mathbf{N}$ be an arbitrary $n \times n$, say, real, *anti-symmetric* (or skew-symmetric) matrix. Are the following two (2) matrices

$$\mathbf{A} \equiv (\mathbf{1} + \mathbf{N})^{-1} (\mathbf{1} - \mathbf{N}),$$

and
$$\mathbf{B} \equiv (\mathbf{1} + \mathbf{N}) (\mathbf{1} - \mathbf{N})^{-1} \qquad \text{(a)}$$

orthogonal ? How about the matrices:

$$\mathbf{C} \equiv (\mathbf{1} - \mathbf{N})^{-1} (\mathbf{1} + \mathbf{N}),$$

and
$$\mathbf{D} \equiv (\mathbf{1} - \mathbf{N}) (\mathbf{1} + \mathbf{N})^{-1}, \qquad \text{(b)}$$

i.e. are they orthogonal ? Proof required, no guesses or "hand-waving" arguments.

Remarks: Here, the following notations are employed:

$\mathbf{1} = n \times n$ *unit* (diagonal) matrix;

$\mathbf{M}^T$ is the *transpose* of (an arbitrary matrix) $\mathbf{M}$,

and $\mathbf{M}^{-1}$ is the *inverse* of $\mathbf{M}$.

Further, it can be shown that $(\mathbf{1} + \mathbf{N})$ is never singular, i.e. *you can assume that* $(\mathbf{1} + \mathbf{N})^{-1}$ *always exists.*

Problem 3

It is known that the potential $u(x, y)$ inside a rectangle with corners at the points (0,0), (a,0), (0,b) and (a,b) satisfies the two dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

If the potential on the boundary of the rectangle is given as

$u(x,0) = x,\ \ a > x \geq 0,$
$u(x,b) = 0,\ \ a \geq x \geq 0,$
$u(0, y) = 0,\ \ b \geq y \geq 0,$
$u(a, y) = 0, b \geq y \geq 0$

Find the potential distribution inside the rectangle.

Problem 4

The curve $y = \cos^2(x)$ is plotted on the graph below. Find the slope of the straight line that goes through the origin and is tangent to the $\cos^2(x)$ curve near the second peak as shown in the plot.