Problem 1.

a) Suggest two different methods to compute or approximate the exponential function of a square matrix, e^{At}. Why is this particular function important?

b) Compute e^{At} for the square matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

c) Which of the two methods from part a) would you use to compute e^{At} for $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$, and why?

Problem 2:

Part1: Derive relevant algebraic equations and outline (list key steps) for (1) explicit and (2) implicit approaches to solving the following initial value problem for 2nd order ODE using finite difference approximations with non-uniform time steps.

$$\frac{d^2y}{dt^2} = y \quad \text{with initial conditions: } y(t=0) = y_0 \;\&\; \frac{dy}{dt}(t=0) = y_0^{'}$$

Part 2: Comment whether or not either of these two approaches (and explain why using relevant equations) could be used for solving the initial value problem for 2nd order ODE modified as follows:

$$\frac{d^2y}{dt^2} = y^2 \quad \text{with initial conditions: } y(t=0) = y_0 \;\&\; \frac{dy}{dt}(t=0) = y_0^{'}$$

Part 3: Comment whether or not either of these two techniques (and briefly explain why) could be used for solving the boundary value problem for the following 2nd order ODE:

$$\frac{d^2y}{dt^2} = y \quad \text{with initial conditions: } y(t=0) = y_0 \;\&\; \frac{dy}{dt}(t=T) = y_0^{'}$$

Problem 3:

We use the prime notation (‘) to denote differentiation with respect to x.

Assume that a physical problem is described by the following equation:

$$ay''+by'+cy=d\sin(2x)$$

The following initial (or boundary) conditions are used:

$$y(0)=2$$

$$y'(0)=1$$

Consider 3 cases I.1, I.2 and case II and solve for each case. You are allowed to use the table with Laplace transform pairs as given below and are expected to use the Laplace transform to solve case II.

<u>Case I.1</u>: d=0 and assume that the equation $ar^2 + br + c = 0$ has two different real roots, namely $r=r_1$ and $r=r_2$.

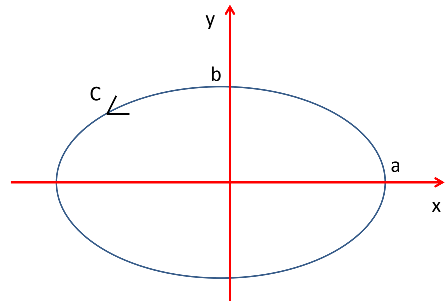
<u>Case I.2</u>: specify the general solution you have found in case I.1 for a=1, b=5 and c=6

<u>Case II</u> : d=1, a=1, b=0, c=1.

Table with Laplace Transform pairs (can be used whenever you believe useful):

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. t^n, n = positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, n = positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$

Problem 4.



(a) Let C be any simple curve bounding a region in the x-y plane having area A. Show that:

$$A = \frac{1}{2}\oint x\,dy - y\,dx$$

(b) Use this result to find the area bounded by the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t < 2\pi$